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A nonlinear unsteady one-dimensional theory for wings in extreme ground effect

By E. O. TUCK

Applied Mathematics Department, University of Adelaide

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Flow induced by a body moving near a plane wall is analysed on the assumption that the normal distance from the wall of every point of the body is small compared to the body length. The flow is irrotational except for the vortex sheet representing the wake. The gap-flow problem in the case of unsteady motion is reduced to a nonlinear first-order ordinary differential equation in the time variable. In the special case of steady flow, some known results are recovered and generalized. As an illustration of the unsteady theory, the problem is solved of a flat plate falling toward the ground under its own weight, while moving forward at uniform speed.

1. Introduction

Fluid-dynamic problems involving bodies moving close to walls are of interest in many different contexts, and there is a considerable literature dealing with such problems. The present paper concerns itself with those ground-effect phenomena which can be approximated as inviscid (high Reynolds number) and incompressible (low Mach number).

The classical inviscid problem is of course the aerodynamics of wings near the ground, and indeed the present work has its most obvious application in that area. The régime of interest in the present paper is, however, that of *very small* clearances to the ground. The early work on aerodynamic ground effect (e.g. as surveyed by Pistolesi 1937), views all effects of the ground as small perturbations to the infinite-fluid flow about the wing. The formal requirement for this to be valid is that the clearance between body and ground be large compared to *all* length scales of the body.

An intermediate régime, in which the flow is qualitatively similar to that in an infinite fluid, but where ground effects are '$O(1)$' perturbations, is when the clearance is comparable to some body dimension. For example (e.g. Bagley 1961; Tuck & Newman 1974), the problem of a thin airfoil in steady ground effect at clearances comparable to the chord (thus large compared to the thickness and/or camber), leads to a singular integral equation for the bound vorticity, whose kernel reduces to the classical lifting-surface kernel for an unbounded fluid, as the clearance/chord ratio tends to infinity. On the other hand, if one lets the clearance/chord ratio tend to *zero* in such a theory, the integral equation 'collapses', and one obtains an almost-trivial explicit result for the loading, in terms of the local distance from the wall.

The small-gap régime is defined formally as that in which the clearance is small compared to the horizontal length scale. For non-thin bodies (e.g. automobiles), a general approach to this class of problem is outlined by Tuck (1975); however, as with any bluff-body flow, little progress can be made without introducing empirical

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assumptions regarding the wake. We assume here, that, in addition, the body is thin, i.e. that not only its lower surface, but also its upper surface, is close to the wall.

Some literature does exist on this small-gap problem. For example, Strand, Royce & Fujita (1962) and others (see references in Gallington & Miller 1970) have noted the 'hydraulic' or channel-flow character of the tightly-constrained flow between the body and the wall. Widnall & Barrows (1970) provided a complete asymptotic solution for the steady-flow case, assuming in addition that the thickness and camber (and 'angle of attack times chord') are small compared to the clearance. The present paper can be considered as an extension of the work of Widnall & Barrows to include 'nonlinear' (thickness, etc., comparable to clearance) and unsteady effects. Unsteadiness due to ground irregularity was included in the linear theory by Barrows & Widnall (1970).

Although the potential applications are to very practical problems, such as large aircraft in ground effect, modern racing cars, tracked ground transportation vehicles (Barrows 1971), and ship manoeuvring in shallow water near banks (Norrbin 1974; Tuck 1978*a*), in the present paper we give only simplified illustrations, involving two-dimensional flow, and specialized geometry and motions. The small-clearance assumption implies that a two-dimensional flow becomes one-dimensional in the gap, and is thus describable by the one-dimensional continuity equation. For a given body under-surface, this equation is a second-order ordinary differential equation for the velocity potential, as a function of the horizontal co-ordinate x along the wall, which can be solved explicitly. Two boundary conditions are needed, and these must come from matching with the outer flow passing over the top of the body. In the appendix, we show that the proper conditions are continuity of velocity potential at the leading edge, and (for bodies without stern appendages) of pressure at the trailing edge.

When these conditions are applied in unsteady flow to determine the net flux exiting from the gap at the stern, the result is a nonlinear ordinary differential equation of the first order with respect to time. In the special case of steady flow, the appropriate solution of this equation is vanishing flux, in a frame of reference fixed in the fluid at infinity. That is, in a frame of reference fixed in the body, the trailing-edge velocity is equal to the free-stream velocity. The present theory then reduces to a nonlinear equivalent of that of Widnall & Barrows (1970). Some nonlinear consequences for the steady lift force are discussed in § 4.

The unsteady problem solved in § 5 is free fall under its own weight of a flat plate toward a plane wall, combined with a horizontal motion at constant speed U. For $U = 0$, a symmetric solution was presented by Yih (1974), in which the plate never actually hits the wall, but ultimately approaches it with an exponential decay of height with time. We find a similar result, but only for U *above* a certain critical value. For lower values of U, the approach to the wall is still gentle, but there is impact at a finite time, with a cubic decay of height with time near impact. It should be noted, however, that we retain the asymmetry between leading and trailing edges even when U is small, and Yih's symmetric solution is likely to be more realistic for small U. We also assume that the angle of attack remains zero for all time, and (at least when $U \neq 0$) this makes the application to problems such as stacking of glass plates (Yih 1974) or the 'sliding of sheets of paper' illustration in G. I. Taylor's ciné film (1967) not yet complete. It would appear, however, that there is no need to invoke viscous effects to explain a number of air-lubrication phenomena.

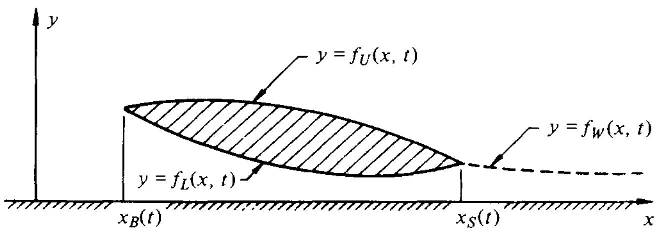

FIGURE 1. Sketch of flow and geometry.

2. Problem formulation

We assume two-dimensional irrotational flow of an incompressible fluid, generated by movement of a thin airfoil-like body as sketched in figure 1, with upper surface $y = f_U$, and lower surface $y = f_L$, i.e. occupying the region

$$f_L(x,t) < y < f_U(x,t), \quad x_B(t) < x < x_S(t). \tag{2.1}$$

The leading edge or bow is at $x = x_B$, and the trailing edge or stern at $x = x_S$. The body has length

$$2l = x_B - x_S, \tag{2.2}$$

which can in the most general case depend on time, but is normally considered to be constant.

The body is everywhere close to a plane boundary at $y = 0$, i.e.

$$f_L, f_U = O(\epsilon l), \tag{2.3}$$

where ϵ is a small parameter. This also requires the body to be *thin*, as in the thin-airfoil theory. However, we do not assume that its thickness $f_U - f_L$ is small compared to the wall separation, e.g. to the minimum value of f_L.

Our task is to solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \tag{2.4}$$

for the velocity potential $\phi(x, y, t)$, subject to suitable boundary conditions. At infinity we have a state of rest, i.e.

$$\phi, \nabla\phi \to 0 \quad \text{as} \quad x, y \to \infty. \tag{2.5}$$

The wall $y = 0$ is impermeable, i.e.

$$\phi_y(x, 0, t) = 0. \tag{2.6}$$

The boundary condition on the moving body surface,

$$\phi_y = f_t + \phi_x f_x, \tag{2.7}$$

applies both with $f = f_L$ and $f = f_U$. Note that when we allow ϵ to become small, we shall not only assume that $f = O(\epsilon)$, but also that $f_x, f_t = O(\epsilon)$.

Finally we must use an appropriate wake and Kutta condition. The trailing edge $x = x_S$ is assumed to shed vortices which remain behind the body in a vortex sheet, with equation

$$y = f_W(x, t), \tag{2.8}$$

the function $f_W(x,t)$ being unknown. The kinematic boundary condition across this surface is given again by (2.7), with $f = f_W$. The dynamic condition is continuity of pressure across the sheet. Thus if $p(x,y,t)$ is the excess of pressure over the value at infinity, then from Bernoulli's equation

$$p = -\rho(\phi_t + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2) \tag{2.9}$$

everywhere in the fluid. The dynamic boundary condition on the wake is

$$p(x, f_W + 0, t) = p(x, f_W - 0, t). \tag{2.10}$$

Equation (2.10) holds in the wake for $x > x_S(t)$ and also holds at the trailing edge $x = x_S(t)$, where it defines the Kutta condition, and ultimately determines the circulation about the body.

In the following section, the approximate solution in the most important region near the wall is developed using intuitive arguments. A complete systematic asymptotic expansion procedure is provided in the appendix.

3. One-dimensional theory

We now assume that in the limit as $\epsilon \to 0$, $\phi = O(\epsilon)$ except in the near-wall 'gap & wake' region G & W defined by

$$0 < y < f(x,t), \quad x > x_B(t), \tag{3.1}$$

where

$$f(x,t) = \begin{Bmatrix} f_L(x,t), & x_B(t) < x < x_S(t) \\ f_W(x,t), & x > x_S(t). \end{Bmatrix} \tag{3.2}$$

In the region G & W, there is a flow of a magnitude which does *not* tend to zero as $\epsilon \to 0$. If we are interested only in leading-order estimates of forces, etc., we can therefore concentrate attention on the flow in region G & W, assuming in effect that $\phi \equiv 0$ when we are outside G & W. A formal justification for this assumption is provided in appendix A by matching techniques

In fact, the flow in G & W is classical one-dimensional or channel flow, in which the fluid moves predominantly in the x direction, parallel to the wall. For example it is consistent with this approximation to use the Taylor expansion

$$\phi = \phi(x,0,t) - \tfrac{1}{2}y^2\phi_{xx}(x,0,t) + \ldots, \tag{3.3}$$

which guarantees satisfaction of (2.4) and (2.6) and satisfies (2.7) if

$$f_t + (f\phi_x)_x = 0. \tag{3.4}$$

This equation is derived in a more-formal manner in appendix A. Alternatively, we may recognize (3.4) as the ordinary one-dimensional continuity equation, expressing conservation of mass in a flow with dominant velocity $\phi_x(x,0,t)$, in a channel of width f which varies both in space and time. From now on, we write $\phi(x,t)$ for $\phi(x,0,t)$.

One integration of (3.4) with respect to x gives

$$f(x,t)\,\phi_x(x,t) = q(t) + \int_x^{x_S(t)} f_t(\xi,t)\,d\xi, \tag{3.5}$$

where

$$q(t) = f(x_S(t),t)\,\phi_x(x_S(t),t) \tag{3.6}$$

is the (so-far unknown) net flux through the gap at the stern. Our primary task is to determine this quantity $q(t)$. A further integration gives the potential ϕ itself. In performing this integration step, we match with the exterior flow $\phi \equiv 0$ outside G & W, by applying the boundary condition

$$\phi(x_B(t), t) = 0, \tag{3.7}$$

at the leading edge $x = x_B(t)$. This is justified formally in appendix A. Thus

$$\phi(x,t) = q(t)\,A(x,t) + B(x,t), \tag{3.8}$$

where

$$A_x = 1/f,$$

i.e.

$$A(x,t) = \int_{x_B(t)}^{x} \frac{d\xi}{f(\xi,t)}, \tag{3.9}$$

and

$$B_x = \frac{1}{f}\int_x^{x_S} f_t\, d\xi,$$

i.e.

$$B(x,t) = \int_{x_B(t)}^{x} \frac{d\xi}{f(\xi,t)} \int_{\xi}^{x_S(t)} f_t(\xi^*, t)\, d\xi^*. \tag{3.10}$$

For a given body lower surface $f = f_L$, (3.8) determines the flow in the gap region G beneath the body, providing we can find $q(t)$. In order to find $q(t)$, we must make use of the Kutta condition (2.10) at the trailing edge $x = x_S(t)$. Now in G & W, we can simplify the Bernoulli equation (2.9) since $\phi_y = O(\epsilon)$, and hence the pressure in G & W is given by

$$p(x,t) = -\rho[\phi_t(x,t) + \tfrac{1}{2}(\phi_x(x,t))^2] + O(\epsilon). \tag{3.11}$$

Since $\phi = O(\epsilon)$ outside G & W, and hence $p = O(\epsilon)$, continuity of p across the wake simply requires that the $O(1)$ dominant part of p vanish in W, i.e.

$$\phi_t(x,t) + \tfrac{1}{2}(\phi_x(x,t))^2 = 0, \quad x > x_S(t). \tag{3.12}$$

A more-formal derivation of (3.12) is given in appendix A. Equation (3.12) is a first-order nonlinear partial differential equation to determine the flow in the wake region W, equivalent to the homogeneous Euler equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3.13}$$

for the velocity $u = \phi_x$. Once (3.12) or (3.13) is solved, we have another first-order equation, (3.4), to solve for the unknown wake boundary $y = f_W(x,t)$. However, it is not necessary to go this far, if we are primarily interested only in the flow beneath the body.

So long as the gap G and wake W regions are essentially continuous with each other (i.e. not interrupted by an edge region where ϕ_y is significant, exceeding $O(\epsilon)$), equation (3.12) can also be applied 'at' $x = x_S(t)$, and is the required Kutta condition to determine the trailing-edge flux $q(t)$. If the under-surface at the trailing edge is 'shaped', or if there is an appendage such as a rudder at a finite angle of attack, no such conclusion can be drawn; some such extensions of the present theory are considered in Tuck (1979).

On substitution of the representation (3.8) for the velocity potential ϕ into the Kutta condition (3.12) at $x = x_S(t)$, we obtain the nonlinear first-order ordinary differential equation

$$A(x_S, t)\,\dot{q} + \dot{A}(x_S, t)\,q + \dot{B}(x_S, t) + \frac{q^2}{2f^2(x_S, t)} = 0. \tag{3.14}$$

Note that the coefficients $\dot{A}$, $\dot{B}$ are obtained by differentiating the corresponding expressions (3.9), (3.10) with respect to time at fixed x, and then setting $x = x_S(t)$; in particular $\dot{A}(x_S, t) \neq d[A(x_S, t)]/dt$. The differential equation (3.14) must be solved in any special case, subject to a suitable initial condition on the flux $q(t)$ at $t = 0$.

4. Steady flow

An important special case is when $f_L(x,t) = f(x+Ut)$, $|x+Ut| \leqslant l$. That is, the rigid body with lower surface $y = f(x+Ut)$ is moving with constant velocity U in the negative x direction. Although it is somewhat easier to solve this problem directly in a moving co-ordinate system, in which the body appears fixed in a uniform stream of magnitude U in the $+x$ direction, we provide the solution here by specialization of the general unsteady results.

We observe that

$$A = \int_{-l}^{x+Ut} \frac{dX}{f(X)}, \tag{4.1}$$

and

$$B = Uf(l)\,A - (x+Ut+l)\,U, \tag{4.2}$$

and find that $\dot{B} = 0$ at the stern. Hence the differential equation (3.14) possesses the solution $q = 0$, which simply means that free-stream conditions prevail at the stern. Equation (3.8) then shows that (4.2) is in fact an expression for the velocity potential $\phi(x,t) = B(x,t)$ everywhere in the gap, from which we can compute (using (3.11)) the pressure in the gap.

$$p = \tfrac{1}{2}\rho U^2\left[1 - \frac{f^2(l)}{f^2(x+Ut)}\right]. \tag{4.3}$$

Since we have assumed that $p = O(\epsilon)$ on the *upper* surface of the body, the net upward force per unit span is

$$F_L = \rho U^2 l(1 - \overline{\lambda^{-2}}) + O(\epsilon), \tag{4.4}$$

where

$$\lambda(x) = f(x)/f(l) \tag{4.5}$$

is the gap thickness, scaled with respect to that at the trailing edge, and a bar denotes a mean value over the length of the body, i.e.

$$\overline{\lambda^{-2}} = \frac{1}{2l}\int_{-l}^{l} \frac{dx}{(\lambda(x))^2}. \tag{4.6}$$

The result (4.4) is equivalent to one obtained by Strand, Royce & Fujita (1962).

It is instructive to compare the above lift force with that obtained by use of the Kutta–Joukowski theorem, which applies only in unbounded media. Thus (since ϕ is continuous elsewhere), the net circulation Γ around the body is generated entirely by the jump in the velocity potential ϕ, from its value $\phi(x_S, t)$ immediately below the trailing edge, to its $O(\epsilon)$ value immediately above it. Hence, to leading order,

$$\begin{aligned} \Gamma &= \phi(x_S, t) \\ &= -2Ul + Uf(l) \int_{-l}^{l} \frac{dx}{f(x)} \\ &= -2Ul(1-\overline{\lambda^{-1}}). \end{aligned} \tag{4.7}$$

Thus the Kutta–Joukowski lift is

$$\begin{aligned} F_K &= -\rho U \Gamma \\ &= 2\rho U^2 l(1-\overline{\lambda^{-1}}). \end{aligned} \tag{4.8}$$

In cases where the body's thickness or camber is small compared to the wall separation, the Kutta–Joukowski theorem becomes asymptotically valid, and $F_K \to F_L$, since in that limit $(f_0(x) \simeq f_0(l))$

$$\begin{aligned} 1-\overline{\lambda^{-2}} &\to 2(1-\overline{\lambda^{-1}}) \\ &\to 2(\overline{\lambda}-1). \end{aligned} \tag{4.9}$$

That is,

$$F_L \to 2\rho U^2 l(\overline{\lambda}-1), \tag{4.10}$$

which depends only on the net area lying between the lower surface of the body and a horizontal line through its trailing edge.

For example, if the body is a flat plate at an angle of attack α, then this linearized approximation predicts that

$$F_L = \tfrac{1}{2}\rho U^2 \, 2l\alpha \left(\frac{2l}{f(l)}\right). \tag{4.11}$$

The result (4.11) was found by Widnall & Barrows (1970). The full nonlinear result (4.4) for this case is

$$F_L = \tfrac{1}{2}\rho U^2 \, 2l\alpha \left(\frac{2l}{f(l)+2l\alpha}\right), \tag{4.12}$$

which is inversely proportional to the *leading-edge* clearance at fixed angle of attack.

It is possible (Tuck 1978*b*) to use the present unsteady theory to analyse the transient development of the steady flow and forces, e.g. subsequent to an impulsive start from a state of rest. The results indicate a significant degree of non-linearity in the transient flow, and in particular the approach to the steady solution is qualitatively different for negative and positive angles of attack.

5. Falling flat plate in steady horizontal motion

We now assume $f_L(x, t) = y_0(t)$, thereby allowing an arbitrary vertical movement for a body with a flat undersurface that remains parallel to the ground. We assume uniform horizontal translation to the left at speed U, i.e. set $x_B(t) = -l - Ut$, $x_S(t) = +l - Ut$, with U constant. The velocity potential (3.8) becomes

$$\phi = (x+Ut+l)\,u + \tfrac{1}{2}(x+Ut+l)\,(x+Ut-3l)\,\beta, \tag{5.1}$$

where

$$u = q/y_0 = \phi_x(-Ut+l,t) \tag{5.2}$$

is the gap velocity at the stern, and

$$\beta = -\dot{y}_0/y_0. \tag{5.3}$$

The differential equation (3.14) becomes

$$2l\dot{u} - 2l^2\dot{\beta} + Uu + \tfrac{1}{2}u^2 = 0. \tag{5.4}$$

For any *given* time history of vertical movement $y_0(t)$, we must evaluate the quantity $\beta(t)$, and then solve the differential equation (5.4) for $u(t)$.

The pressure on the plate's under surface is given by

$$\begin{aligned}-p/\rho = {} & (x+Ut+l)\,\dot{u}+\tfrac{1}{2}(x+Ut+l)\,(x+Ut-3l)\,\dot{\beta}+U[u+\beta(x+Ut-l)] \\ & +\tfrac{1}{2}[u+\beta(x+Ut-l)]^2,\end{aligned} \tag{5.5}$$

and vanishes as required at the stern $x+Ut = l$, if (5.4) is satisfied. The net upward force is

$$F_L = -\rho l[2l\dot{u}-3l^2\dot{\beta}+2U(u-\beta l)+(u-\beta l)^2+\tfrac{1}{3}l^2(\dot{\beta}+\beta^2)] \tag{5.6}$$

and the (bow-down) moment is

$$M = -\tfrac{2}{3}\rho l^3[\dot{u}-l\dot{\beta}+(U+u)\,\beta-l\beta^2]. \tag{5.7}$$

For example, consider a freely-falling plate of mass m per unit span, constrained against rotation. Then $y_0(t)$ itself is determined by solving the equation of motion

$$F_L = m(\ddot{y}_0+g), \tag{5.8}$$

i.e.

$$2l\dot{u}-\tfrac{8}{3}l^2\dot{\beta}+2Uu+u^2-2(U+u)\,\beta l+\tfrac{4}{3}l^2\beta^2 = -\frac{m}{\rho l}(\ddot{y}_0+g). \tag{5.9}$$

Equations (5.3), (5.4) and (5.9) are a set of 3 coupled nonlinear first-order differential equations to solve for u, β and y_0.

If we define $t = t_0$ as the instant at which the plate hits the wall, so that motion takes place for $t \to t_0$, then so long as t_0 remains finite, the final stage of the fall appears to be described by the estimates

$$u = -\frac{2l}{t-t_0}-U-\left[\frac{mg}{4\rho l^2}+\frac{U^2}{6l}\right](t-t_0)+O(t-t_0)^2, \tag{5.10}$$

$$\beta = \frac{-3}{t-t_0}-\frac{U^2(t-t_0)}{4l}+O(t-t_0)^2, \tag{5.11}$$

and

$$y_0 = -c(t-t_0)^3\left[1+\frac{U^2(t-t_0)^2}{8l^2}+O(t-t_0)^3\right], \tag{5.12}$$

for some constant c. This contrasts with the exponential approach to the wall found by Yih (1974). Yih solved the corresponding problem at $U = 0$, by making the empirical assumption that $p = 0$ at *both* edges of the plate. This means that vortex sheets must spring from both edges, there being of course no distinction between 'leading' and 'trailing' edges if $U = 0$. Although the result is different, the conclusion that the approach to the wall is relatively 'gentle' still holds. However, the present requirement of no rotation is rather restrictive, since the limiting bow-down moment is

$$M = \frac{8}{3}\frac{\rho l^4}{(t-t_0)^2} + O(1). \tag{5.13}$$

In fact it is possible to have an exponential approach to the wall, for *sufficiently-large* horizontal speed U. Thus, the system (5.3), (5.4), (5.9) allows a solution with $u \to 0, \beta \to \beta_0$, and

$$y \to c e^{-\beta_0 t} \to 0, \quad t \to \infty, \tag{5.14}$$

where β_0 is a positive constant determined from the quadratic equation

$$-2Ul\beta_0 + \tfrac{4}{3}l^2\beta_0^2 = -mg/\rho^l, \tag{5.15}$$

i.e.

$$\beta_0 = \frac{3U}{4l} \pm \left(\frac{9U^2}{16l^2} - \frac{3mg}{4\rho l^3}\right)^{\frac{1}{2}}. \tag{5.16}$$

This solution possesses a finite bow-up limiting moment M.

The exponentially-decaying solution can exist only if the square root in (5.16) is real, i.e. if

$$U > \left(\frac{4mg}{3\rho l}\right)^{\frac{1}{2}} = U_0. \tag{5.17}$$

The value of β_0 at this limiting speed U_0 is one half of the value found by Yih at $U = 0$, and as $U \to \infty$, the dominant (smaller β_0) decaying term has $\beta_0 \to 0$. The present theory must break down for sufficiently small U, and Yih's theory appears a suitable extrapolation to $U = 0$, with the advantage of flow symmetry about the centre of the plate.

The above system was solved numerically for motion starting from rest at $y_0 = h$. We use a non-dimensional version of the equations, namely

$$\dot{y}_0 = -\beta y_0, \quad y(0) = 1,$$

$$\dot{u} = \dot{\beta} - \tfrac{1}{2}Uu - \tfrac{1}{4}u^2, \quad u(0) = 0,$$

$$\dot{\beta} = \frac{-\frac{1}{2}Uu + U\beta - \frac{1}{4}u^2 + u\beta - \frac{2}{3}\beta^2 - \frac{1}{2}my_0\beta^2 - m}{-\frac{1}{3} - \frac{1}{2}my_0}, \quad \beta(0) = 0, \tag{5.18}$$

where the scale for time is the time taken to fall a distance h in a vacuum, i.e.

$$T = \left(\frac{2h}{g}\right)^{\frac{1}{2}}, \tag{5.19}$$

and y_0 is scaled with h, u and U with l/T, β with $1/T$, and m with $\rho l^3/h$.

It appears from these computations that whenever the condition (5.17) is satisfied (i.e. at high speed U), the exponential-decay solution holds, with the smaller of the two values of β_0 in (5.16). On the other hand, if (5.17) is not satisfied (low speed U), the body does indeed hit the ground at a finite time t_0, and $t_0 \to \infty$ as $U \to U_0$.

Figure 2 shows a typical set of numerical results at (scaled) $m = 1$, for various values of (scaled) U. The critical scaled speed is $U_0 = (\frac{8}{3})^{\frac{1}{2}} = 1{\cdot}64$ in this case. For example, we find at $U = 1$ that the body hits the ground (with zero velocity and acceleration as predicted by (5.12)) at $t \simeq 3{\cdot}3$, whereas at $U = 2$ there is an exponential approach as in (5.14), with $\beta_0 \simeq 0{\cdot}63$, as predicted by (5.16). The curve with $U = 0$

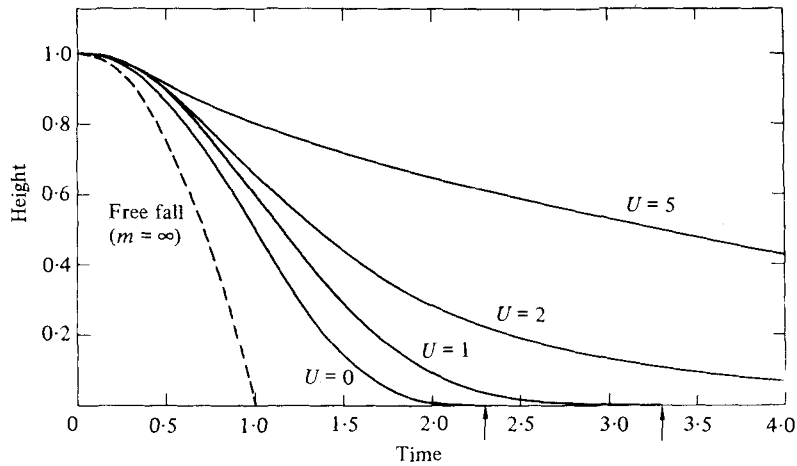

FIGURE 2. Scaled trajectories of height versus time for fall of a plate of (scaled) mass $m = 1$, for various (scaled) horizontal speeds U. Arrows indicate times of impact for $U = 0, 1$; there is no impact if $U > 1{\cdot}64$.

has no physical significance but is included as a limiting case for small U. Figure 2 also shows (dashed) the parabolic free fall in a vacuum, equivalent to (scaled) $m \to \infty$ for all finite U. Fluid-dynamic effects always postpone the impact (sometimes forever!), and also make the impact gentle if it does occur.

Appendix A. Formal asymptotic development

The flow region is divided into five separate asymptotic regions as $\epsilon \to 0$, namely:

(1) *Exterior* E: $y/\epsilon \to +\infty$.

(2) *Gap* G: $y = O(\epsilon)$, $\dfrac{x-x_B}{\epsilon}$, $\dfrac{x_S-x}{\epsilon} \to +\infty$.

(3) *Wake* W: $y = O(\epsilon)$, $\dfrac{x-x_S}{\epsilon} \to +\infty$.

(4) *Bow* B: $x-x_B$, $y = O(\epsilon)$.

(5) *Stern* S: $x-x_S$, $y = O(\epsilon)$.

In each region, as $\epsilon \to \infty$ we make a separate asymptotic expansion of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \ldots, \tag{A 1}$$

where, when necessary, we identify the region by a superscript, e.g. the exterior expansion is

$$\phi = \phi_0^E + \phi_1^E + \phi_2^E + \ldots. \tag{A 2}$$

The subscript indicates an ordering with respect to ϵ. In principle, we only need require that $\phi_{n+1} = o(\phi_n)$ as $\epsilon \to 0$; however, in the present problem it soon becomes clear that all expansions are power series in ϵ, i.e. that $\phi_n = O(\epsilon^n)$. It is not necessary that all terms be present, and in particular we shall find that $\phi_0^E \equiv \phi_0^B \equiv 0$. Note that we are making the reasonable implicit assumption that in no region do flow quantities tend to infinity as $\epsilon \to 0$. We also assume that time differentiation does not alter the order of magnitude of any term. We now consider each region in turn.

(1) *Exterior region*

The full Laplace equation (2.4) holds for all terms of the exterior expansion, as does the boundary condition (2.5) at infinity. The boundary $y = f(x,t) = O(\epsilon)$ shrinks to the wall $y = 0_+$, as seen by an exterior observer, and the boundary condition (2.7) yields limiting boundary conditions on $y = 0_+$ for successive terms, namely

$$\phi^E_{0y} = 0,$$

$$\phi^E_{1y} = f_t + (f\phi^E_{0x})_x, \tag{A 3}$$

etc. These hold for $x_B < x < x_S$ with $f = f_U$, and for $x > x_S$ with $f = f_W$; for $x < x_B$ all terms satisfy $\phi^E_{ny} = 0$ on $y = 0$.

The boundary-value problem for ϕ^E_0 is homogeneous, and has the unique solution $\phi^E_0 \equiv 0$, unless there is any contribution from singular points at $(x_B, 0)$, $(x_S, 0)$ that model the edge regions, as seen by an exterior observer. Such a contribution can only occur if there are apparent singularities, notably sources or sinks, of $O(1)$ strength as $\epsilon \to 0$. Since the gap within which any such flux must squeeze is $O(\epsilon)$, this can only happen if the edge and gap velocities are $O(\epsilon^{-1})$ and thus tend to infinity as $\epsilon \to 0$, a possibility we have excluded. Hence we assume that any apparent singularities at $(x_B, 0)$, $(x_S, 0)$ are at most $O(\epsilon)$, and affect only ϕ^E_1.

Thus ϕ^E_0 satisfies uniformly-homogeneous boundary conditions for Laplace's equation, and hence vanishes identically. The exterior flow is therefore $O(\epsilon)$, and its leading term satisfies

$$\phi^E_{1y}(x, 0_+) = f_t(x,t), \tag{A 4}$$

together with suitable matching conditions describing possible $O(\epsilon)$ singularities at the bow $(x_B, 0)$. Although in principle a singularity is also feasible at the stern $(x_S, 0)$, this is ultimately excluded by the Kutta or smooth detachment condition.

(2) *Gap region*

In the gap (and wake) region, the y co-ordinate must be stretched relative to the x co-ordinate. In effect, we have $y = O(\epsilon)$, $\partial/\partial y = O(\epsilon^{-1})$, and there is an $O(\epsilon^2)$ imbalance between the two terms in the Laplace equation (2.4). Thus, the successive terms in the expansion for ϕ satisfy

$$\phi^G_{0yy} = 0, \quad \phi^G_{1yy} = 0, \quad \phi^G_{2yy} = -\phi^G_{0xx}, \tag{A 5}$$

etc. The general solutions of (A 5) contain terms linear in y that must vanish, since $\phi^G_{ny} = 0$ at $y = 0$ for all n. Thus we may take as our solutions

$$\phi^G_0 = \phi^G_0(x,t), \quad \phi^G_1 = \phi^G_1(x,t), \quad \phi^G_2 = \phi^G_2(x,0,t) - \tfrac{1}{2}y^2\phi^G_{0xx}(x,t), \tag{A 6}$$

etc.

The limiting forms of the boundary condition (2.7) on $y = f = f_L(x,t)$ are

$$\phi^G_{0y} = 0, \quad \phi^G_{1y} = 0$$

and

$$\phi^G_{2y} = f_t + \phi^G_{0x} f_x. \tag{A 7}$$

The first two of these conditions (A 7) are already satisfied, and the last is satisfied if and only if $\phi = \phi_0^G(x,t)$ satisfies the one-dimensional continuity equation (3.4).

(3) *Wake region*

There is no formal mathematical difference between the wake region and the gap region, except for the fact that the bounding surface $y = f = f_W(x,t)$ is unknown. Thus the same solutions (A 6) apply, and (3.4) is still a necessary condition on $\phi = \phi_0^W(x,t)$.

The additional condition that must be considered in order to determine f_W is the jump condition (2.10) across the vortex sheet $y = f_W(x,t)$. The left-hand side of this equation must be evaluated using the exterior-region expansion, and the right-hand side using the wake-region expansion. Thus we have

$$\phi_{1t}^E(x, 0_+, t) + O(\epsilon^2) = \phi_{0t}^W(x,t) + \tfrac{1}{2}(\phi_{0x}^W(x,t))^2 + O(\epsilon). \qquad \text{(A 7)}$$

Since the left-hand side of (A 7) is $O(\epsilon)$ and the terms on the right-hand side are $O(1)$, necessarily $\phi = \phi_0^W(x,t)$ satisfies (3.12) for $x > x_S(t)$.

(4) *Bow region*

It is convenient in discussing this flow region to adopt a special stretched co-ordinate set (X, Y) defined by

$$x = x_B(t) + h_B(t)\, X, \quad y = h_B(t)\, Y, \qquad \text{(A 8)}$$

where

$$h_B(t) = f_L(x_B(t), t) = f_U(x_B(t), t) = O(\epsilon) \qquad \text{(A 9)}$$

is the gap size at the bow. Since x and y are scaled in the same way, the full Laplace equation (2.4) applies with respect to (X, Y), to each term in the bow-region expansion, and each term satisfies $\phi_{nY}^B(X, 0) = 0$.

The boundary condition as $Y \to \infty$ corresponds to matching with the singular limit of the exterior flow as $(x, y) \to (x_B, 0)$. Although we have so far left that question open, it is intuitively clear that the character of this singularity will be that of a *sink* of some $O(\epsilon)$ strength to be determined. It is convenient to write its strength as $2u_B(t) \,.\, h_B(t)$, where $u_B(t)$ is an $O(1)$ velocity to be determined. Thus the boundary condition at infinity can be written

$$\phi \to -\frac{u_B h_B}{\pi}[\log(x^2+y^2)^{\frac{1}{2}}], \quad y \to \infty,$$

and successive terms in the bow-region expansion satisfy:

$$\phi_0^B \to 0, \quad Y \to \infty;$$

$$\phi_1^B \to -\frac{u_B h_B}{\pi}[\log(X^2+Y^2)^{\frac{1}{2}}], \quad Y \to \infty. \qquad \text{(A 10)}$$

An additional boundary condition at 'infinity' is that which applies as $X \to +\infty$, $0 < Y < 1$, corresponding to matching with the gap region, in its limit as $x \to x_{B+}$, namely

$$\phi \to \phi_0^G(x_B, t) + (x - x_B) \,.\, \phi_{0x}^G(x_B, t) + \phi_1^G(x_B, t) + O(\epsilon^2).$$

Hence successive terms in the bow-region series must satisfy

$$\phi_0^B \to \phi_0^G(x_B, t), \qquad \text{(A 11)}$$

$$\phi_1^B \to \phi_1^G(x_B, t) + h_B\, \phi_{0x}^G(x_B, t) \,.\, X, \qquad \text{(A 12)}$$

as $X \to \infty$. This means that in this limit, ϕ_0^B generates zero net velocity, $\phi_{0X}^B \to 0$ as $X \to \infty$, whereas

$$\phi_{1X}^B \to h_B(t) \,.\, \phi_{0x}^G(x_B(t), t). \tag{A 13}$$

The final boundary condition is on the leading-edge tip of the airfoil $y = f(x, t)$, $f = f_L$ and f_U, $x \to x_B$. We assume in this appendix that both f_L and f_U are analytic near $x = x_B$, with $f_L(x_B, t) = f_U(x_B, t)$. This means that the airfoil possesses a sharp leading edge; the case when the leading edge is blunt is treated in Tuck (1979). Now as $x \to x_B$,

$$y = f \to h_B(t) + (x - x_B(t)) f_x(x_B(t), t) + \ldots$$

or

$$Y = 1 + X f_x + O(\epsilon^2)$$

and hence the limiting bow-region airfoil is the semi-infinite flat plate $Y = 1 \pm 0$, $X > 0$. The limiting boundary conditions on that plate follow from (2.7) as

$$\phi_{0Y}^B = 0,$$

$$\phi_{1Y}^B = f_x \,.\, (X \phi_{0X}^B)_X, \tag{A 14}$$

etc.

The boundary-value problem for the leading term ϕ_0^B is homogeneous except for the $X \to +\infty$ limit in (A 11) and the problem for the derivative ϕ_{0X}^B is *entirely* homogeneous. Thus necessarily $\phi_{0X}^B(X, Y) \equiv 0$. But then (A 10) guarantees that $\phi_0^B(X, Y) \equiv 0$, and hence (A 11) indicates that $\phi_0^G(x_B(t), t) = 0$, which is equation (3.7) of the text. The residual $O(\epsilon)$ boundary-value problem for ϕ_1^B can now be solved easily and completely by a Schwartz–Christoffel conformal mapping, as in Widnall & Barrows (1970), but as we have no use for such a solution it is not presented here. We note in passing, however, that the problem for ϕ_1^B is quasi-steady, the time co-ordinate playing a parametric role only, and that continuity implies that $u_B = \phi_{0x}^G(x_B(t), t)$.

(5) *Stern region*

If we adopt a similar set of scaled co-ordinates (X, Y), such that

$$x = x_s(t) + h_s(t) X, \quad y = h_s(t) Y, \tag{A 15}$$

where $h_s(t) = f_L(x_s(t), t) = f_U(x_s(t), t)$ is the stern gap size, again all $\phi_n^s(X, Y)$ satisfy Laplace's equation in $Y > 0$, and $\phi_{nY}^s = 0$ on $Y = 0$. Matching with the gap-region flow requires that

$$\phi_0^s \to \phi_0^G(x_s(t), t), \quad \phi_{0X}^s \to 0,$$

and

$$\phi_1^s \to \phi_1^G(x_s(t), t) + h_s \phi_{0x}^G(x_s(t), t) \,.\, X \tag{A 16}$$

as $X \to -\infty$, $0 < Y < 1$. Similarly, matching with the wake-region flow requires that

$$\phi_0^s \to \phi_0^W(x_s(t), t), \quad \phi_{sX}^0 \to 0,$$

and

$$\phi_1^s \to \phi_1^W(x_s(t), t) + h_s \phi_{0x}^W(x_s(t), t) \,.\, X, \tag{A 17}$$

as $X \to +\infty$, $Y = O(1)$.

The airfoil boundary conditions when $f(x, t)$ is analytic near $x = x_s(t)$ are also similar to those for the bow region, i.e.

$$\phi^s_{0Y} = 0, \quad \phi^s_{1Y} = f_x \cdot (X\phi^s_{0X})_X, \tag{A 18}$$

on $Y = 1 \pm 0$, $X < 0$. Again, non-analytic f is considered in Tuck (1979).

Instead of matching with the exterior region, as in the bow case, now we must find the limiting form of the jump condition (2.10) in the stern region, evaluating the left-hand side using limits as $(x, y) \to (x_s, 0)$ of exterior-region quantities. There is now no reason to suspect singularities, since the Kutta condition demands smooth stern detachment. Therefore we may assume that the leading-order exterior term ϕ^E_1 is an analytic $O(\epsilon)$ quantity near $(x_s, 0)$, and hence so is the left-hand side of (2.10). In the (X, Y) co-ordinate system, this means that

$$\phi_t - \frac{1}{h_s}[(\dot{x}_s + X\dot{h}_s)\,\phi_X + Y\dot{h}_s\,\phi_Y] + \frac{1}{2h_s^2}(\phi_X^2 + \phi_Y^2) = O(\epsilon) \tag{A 19}$$

on $Y = f_W(x, t)/h_s(t) = F(X, t)$, say. In particular, in the limit as $\epsilon \to 0$, the leading-order $O(1)$ term $\phi = \phi^s_0$ must generate zero velocity on the wake boundary, i.e.

$$\phi^{s^2}_{0X} + \phi^{s^2}_{0Y} = 0 \quad \text{on} \quad Y = F.$$

Once again the derivative ϕ^s_{0X} satisfies homogeneous boundary conditions and hence must vanish identically. However, we can now at most assert that ϕ^s_0 is a function of time alone, and can no longer demand that it vanish. Thus we have

$$\phi^s_0 = \phi^s_0(t),$$

where

$$\phi^s_0(t) = \phi^G_0(x_s(t), t) = \phi^W_0(x_s(t), t). \tag{A 20}$$

Thus, we have proved continuity of gap and wake potentials across the trailing edge. In order to verify the form of the Kutta condition (3.12), we must proceed to the next term ϕ^s_1.

Now, on substituting $\phi = \phi^s_0(t) + \phi^s_1(X, Y, t)$ into (A 19), we obtain as the limiting boundary condition for $\phi^s_1 = O(\epsilon)$, the result

$$-\frac{\dot{x}_s}{h_s}\phi^s_{1X} + \frac{1}{2h_s^2}(\phi^{s^2}_{1X} + \phi^{s^2}_{1Y}) = -\frac{d\phi^s_0}{dt}. \tag{A 21}$$

We now observe that all boundary conditions can be satisfied by the solution

$$\phi^s_1(X, Y, t) = \phi^s_1(t) + h_s(t)\,u_s(t)\,X, \tag{A 22}$$

providing

$$\phi^s_1(t) = \phi^G_1(x_s(t), t) = \phi^W_1(x_s(t), t) \tag{A 23}$$

and

$$u_s(t) = \phi^G_{0x}(x_s(t), t) = \phi^W_{0x}(x_s(t), t), \tag{A 24}$$

with

$$F(X, t) = 1, \tag{A 25}$$

and

$$-\dot{x}_s u_s + \tfrac{1}{2}u_s^2 = -d\phi^s_0/dt. \tag{A 26}$$

Physically, this solution simply corresponds to a uniform stream of magnitude $u_s(t)$, the wake boundary being a plane projection of the trailing edge of the airfoil.

We have now shown that both the velocity potential ϕ_0 and its x derivative ϕ_{0x} are continuous across the trailing edge from gap G to wake W. In effect, the stern edge region S is superfluous. Since ϕ_0^W satisfies (3.12), this is sufficient already to confirm validity of (3.12) as the trailing edge condition for ϕ_0^G, but in fact, (A 26) is a specific confirmation of that fact, since

$$\frac{d\phi_0^s}{dt} = \frac{d}{dt}\phi_0^G(x_s(t),t)$$

$$= \phi_{0x}^G \dot{x}_s + \phi_{0t}^G. \qquad \text{(A 27)}$$

Thus (A 26) states directly that

$$\phi_{0t}^G + \tfrac{1}{2}(\phi_{0x}^G)^2 = 0$$

at $x = x_s(t)$, which is the required trailing edge condition (3.12) for the gap potential $\phi = \phi_0^G(x,t)$.

REFERENCES

BAGLEY, J. A. 1961 The pressure distribution on two-dimensional wings near the ground. *Aero. Res. Counc. R. & M.* no. **3238**.

BARROWS, T. M. 1971 Progress on the ram-wing concept, with emphasis on lateral dynamics. *U.S. Dept. Transp. Rep.* DOT-TSC-FRA-71-7.

BARROWS, T. M. & WIDNALL, S. 1970 The aerodynamics of ram-wing vehicles for application to high-speed ground transportation. *A.I.A.A. Paper*, no. 70–142. *8th Aerospace Sci. Meeting, New York.*

GALLINGTON, R. W. & MILLER, M. K. 1970 The ram-wing: a comparison of simple one-dimensional theory with wind tunnel and free flight results. *A.I.A.A. Paper*, no. 70–971. *Guidance, Control and Flight Mech. Conf., Santa Barbara, California.*

NORRBIN, N. H. 1974 Bank effects on a ship moving through a short dredged channel. *10th Symp. Naval Hydro., Cambridge, Mass. Proc., Office of Naval Res., Washington, D.C.*, pp. 71–88.

PISTOLESI, E. 1937 Ground effect-theory and practice. *N.A.C.A., T.M.* no. 828.

STRAND, T., ROYCE, W. W. & FUJITA, T. 1962 Cruise performance of channel-flow ground-effect machines. *J. Aero. Sci.* **29**, 702–711.

TAYLOR, G. I. 1967 *Low-Reynolds-Number Flows.* 16 mm color sound film, Encyclopaedia Britannica Educational Corp., Chicago, Illinois.

TUCK, E. O. 1975 Matching problems involving flow through small holes. *Adv. Appl. Mech.* **15**, 89–158.

TUCK, E. O. 1978*a* Hydrodynamic problems of ships in restricted waters. In *Annual Review of Fluid Mechanics* (ed. Van Dyke, M.) **10**, 33–44. Annual Reviews Inc.

TUCK, E. O. 1978*b* Unsteady Small-Gap Ground Effects, Engineering Science Report 78–1, Calif. Inst. of Tech., Pasadena.

TUCK, E. O. 1979 Applied Mathematics Report T7906, University of Adelaide.

TUCK, E. O. & NEWMAN, J. N. 1974 Hydrodynamic interactions between ships. *10th Symp. Naval Hydro., Cambridge, Mass. Proc., Office of Naval Res., Washington, D.C.*, pp. 35–70.

WIDNALL, S. E. & BARROWS, T. M. 1970 An analytic solution for two- and three-dimensional wings in ground effect. *J. Fluid Mech.* **41**, 769–792.

YIH, C. S. 1974 Fluid mechanics of colliding plates. *Phys. Fluids* **17**, 1936–1940.